Mid-term :

Change of variables:

$$\iint_{\Omega} F(x,y)\, dA(x,y) = \iint_{\Omega} F(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dx dy$$

– Remember there is a sign of absolute value!

– You can use the trick $\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}$

Eg. Evaluate the double integral $\iint_D x^3 dA$

where D is the region bounded by $y=\sqrt{x}$, $y=3\sqrt{x}$, $xy=1$, $xy=4$ $x,y>0$

Ans: there are many different choices of change of variable

e.g. $u = y^2/x$, $v=xy$, then D becomes: $1 \le u \le 9$, $1 \le v \le 4$

$x^3 = v^2/u$

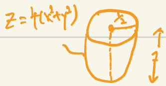
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -\frac{y^2}{x^2} & \frac{1}{x} \\ y & x \end{vmatrix} = -\frac{2y^2}{x} = -2u \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2u}$$

$$\Rightarrow \iint_D x^3 dA = \int_1^9 \int_1^4 \frac{v^2}{u} \cdot \frac{1}{2u}\, dv du$$

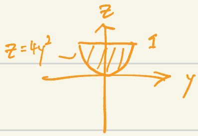
Change the order of integration of the following iterated integral from $dz\,dy\,dx$ to $dx\,dy\,dz$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\sqrt{\frac{1}{4}-x^2}}^{\sqrt{\frac{1}{4}-x^2}}\int_{4(x^2+y^2)}^{1} f(x,y,z)\,dz\,dy\,dx$$

Ans: projection to xy plane:

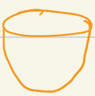


$\Rightarrow$ Project to y-z plane:

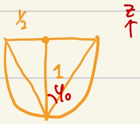


$$\Rightarrow \int_0^1\int_{-\sqrt{\frac{z}{4}}}^{\sqrt{\frac{z}{4}}}\int_{-\sqrt{\frac{z}{4}-y^2}}^{\sqrt{\frac{z}{4}-y^2}} \qquad dz$$

Change the above integral to polar coordinates



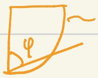
$0 \le \theta \le 2\pi$ always.



for $0\le\varphi\le\varphi_0$

$0\le\rho\le\frac{1}{\cos\varphi}$

for $\varphi_0\le\varphi\le\frac{\pi}{2}$



$z=4r^2 \rightsquigarrow \rho\cos\varphi = 4\rho^2\sin^2\varphi$

$$\rho = \frac{\cos\varphi}{4\sin^2\varphi}$$

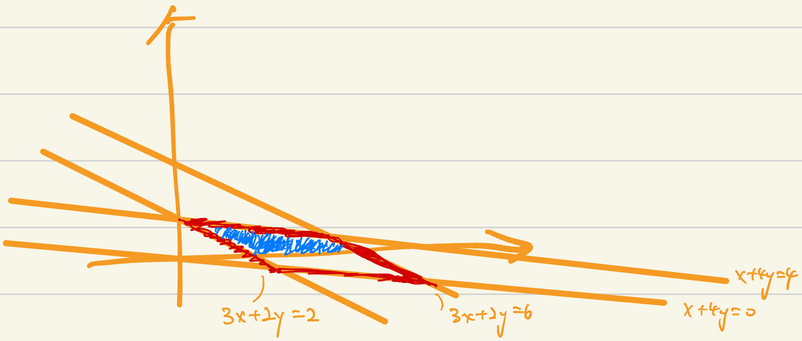
So $\quad 0\le\rho\le\frac{\cos\varphi}{4\sin^2\varphi}$

Homework: Use transformation $u = 3x+2y$, $v = x+4y$ $\left(\Rightarrow x = \frac{1}{5}(2u-v),\ y = \frac{1}{10}(3v-u) \right)$

to evaluate $\iint_R (3x^2+14xy+8y^2)\,dx\,dy$

where R in the quadrant bounded by $2 \le 3x+2y \le 6$

$0 \le x+4y \le 4$



R: $2 \le 3x+2y \le 6$: $2 \le u \le 6$

from $y=0$ to $x+4y=4$

$\frac{1}{10}(3v-u)=0$ $\quad v=4$

$\Rightarrow \int_2^6 \int_{u/3}^4 \quad dv\,du$

line Integral:

Q1: Find $\int_C x\,dy$, where C is the circle centered at (p,q) with radius R (in anti-clockwise direction)

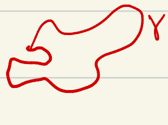
Ans: Step 1: find a parametrization

$$\gamma(t) = (p + R\cos t,\ q + R\sin t) \qquad 0 \le t \le 2\pi$$

$$\begin{aligned}
\text{The integral} &= \int_0^{2\pi} (p + R\cos t)\, d(q + R\sin t) \\
&= \int_0^{2\pi} pR\cos t + R^2\cos^2 t\, dt \\
&= \int_0^{2\pi} pR\cos t + \tfrac{1}{2}R^2(1+\cos 2t)\, dt \\
&= pR\sin t + \tfrac{1}{2}R^2\left(t + \tfrac{1}{2}\sin 2t\right)\Big|_0^{2\pi} \\
&= \pi R^2
\end{aligned}$$

- The integral is independent of parametrization
- Later you will learn that, for a simple closed curve γ

$$\int_\gamma x\,dy = \text{Area of the region bounded by } \gamma.$$



$$\int \vec{F}\cdot d\vec{r} = \int F_1\,dx + F_2\,dy + F_3\,dz \quad , \vec{F} = (F_1, F_2, F_3)$$

Q2: Let C be a smooth curve joining (1,2,3) to (4,5,6)

$\vec{F}(x,y) = yz\vec{i} + xz\vec{j} + xy\vec{k}$

Ans: Let $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ be a parametrization of C $0 \le t \le 1$.

$$\int \vec{F}\cdot d\vec{r} = \int_0^1 (\gamma_2\gamma_3, \gamma_3\gamma_1, \gamma_1\gamma_2)\cdot d(\gamma_1, \gamma_2, \gamma_3)$$

$$= \int_0^1 \gamma_2\gamma_3\,d\gamma_1 + \gamma_3\gamma_1\,d\gamma_2 + \gamma_1\gamma_2\,d\gamma_3$$

$$= \int_0^1 d(\gamma_1\gamma_2\gamma_3)$$

$$= \gamma_1(t)\gamma_2(t)\gamma_3(t)\Big|_{t=0}^{t=1}$$

← fundamental thm of Calculus for integral over $\mathbb{R}$

$$= 4\cdot5\cdot6 - 1\cdot2\cdot3 = 114$$

Alternatively, one can use fundamental thm for line integral

Note $\vec{F} = \nabla f$ where $f(x,y,z) = xyz$

So $\int_A^B \vec{F}\cdot d\vec{r} = \int_A^B \nabla f\cdot d\vec{r} = f(B) - f(A)$

Rmk: It is independent of the curve.

Midterm Q9:

$f_0 = f$, $f_n(x) = \int_0^x f_{n-1}(t)\,dt$, $n \geq 1$. Show that

$$f_n(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t)\,dt \qquad n \geq 1$$

Ans Induction: $n=1$. $f_1(x) = \int_0^x f(t)dt$ is just the definition of f_1

Assume $f_k(x) = \frac{1}{(k-1)!} \int_0^x f(t)dt \qquad k \geq 1$.

then $f_{k+1}(x) = \int_0^x f_k(s)\,ds$

$$= \int_0^x \frac{1}{(k-1)!} \int_0^s (s-t)^{k-1} f(t)\,dt\,ds$$

$$= \frac{1}{(k-1)!} \int_0^x \int_t^x (s-t)^{k-1} f(t)\,ds\,dt$$

$$= \frac{1}{k!} \int_0^x (s-t)^k f(t) \Big|_{s=t}^{s=x} dt$$

$$= \frac{1}{k!} \int_0^x (x-t)^k f(t)\,dt$$